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$\mathbb{Z}_n$ Baxter Model: Critical Behavior

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The $\mathbb{Z}_n$ Baxter Model is an exactly solvable lattice model in the special case of the Belavin parametrization. We calculate the critical behavior of $\text{Prob}_n(\sigma = \omega^k)$ using techniques developed in number theory in the study of the congruence properties of $p(m)$, the number of unrestricted partitions of an integer m.

KEY WORDS: Statistical mechanics; $\mathbb{Z}_n$ Baxter model; order parameters; critical exponents; partition identities.

1. INTRODUCTION

The $\mathbb{Z}_n$ Baxter model with the Belavin parametrization of the Boltzmann weights is an exactly solvable vertex model[(1)] which for $n = 2$ reduces to Baxter's eight vertex model.[(2,3)] The dual to this $\mathbb{Z}_n$ vertex model is a spin model where the spins now "point" to the nth roots of unity. Thus a relevant quantity is the $\text{Prob}_n(\sigma = \omega^k)$ where $\omega = \exp(2\pi i/n)$ and $k = 0,...,$ $n-1$. In Ref. 1 it was shown that in the ferromagnetic regime

$$\text{Prob}_n(\sigma = \omega^k) = \varphi(x) \sum_{m \equiv k \, (\text{mod}\, n)} p(m)\, x^m \tag{1.1}$$

where $p(m)$ is the number of unrestricted partitions of the integer m and

$$\varphi(x) = \prod_{n=1}^{\infty} (1 - x^n) \tag{1.2}$$

The above expression is well-suited to analyze the $x \to 0$ limit where we see that $\text{Prob}_n(\sigma = 1)$ tends to 1 and the remaining probabilities tend to 0. It

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was further shown in Ref. 1 that as $x \to 1$ all the probabilities tend to $1/n$. Thus the variable x is a "temperature-like" variable for the $\mathbb{Z}_n$ model, with $x = 0$ being the completely ordered state and $x = 1$ being the transition to the disordered state. The variable x cannot be directly related to temperature because the Belavin parametrization requires that the Boltzmann weights satisfy certain functional relationships. This follows from the fact that the Belavin parametrization contains only four parameters (counting the overall normalization) to describe n^2 parameters for a general $\mathbb{Z}_n$ vertex model. This seems to be a common deficiency in the solutions thus far found for multistate vertex models. Nevertheless it is of some interest to analyze the "critical behavior" of this model. To define the critical exponent β we write

$$x = \exp(-\pi t), \qquad 0 \leqslant t \leqslant \infty \tag{1.3}$$

and introduce the conjugate variable

$$p = \exp(-\pi/t) \tag{1.4}$$

As $t \to 0$, $x \to 1$ and $p \to 0$. We use p as the "deviation from criticality" variable, and thus we wish to determine β as defined by

$$\mathrm{Prob}_n(\sigma = \omega^k) \sim 1/n + cp^\beta \qquad \text{as } p \to 0 \tag{1.5}$$

The congruence properties of $p(n)$ have been studied in considerable detail starting with Ramanujan who stated that

$$\sum_{n=0}^{\infty} p(5n+4)\, x^n = 5\varphi^5(x^5)/\varphi^6(x) \tag{1.6}$$

Note that with such a result, and using the transformation properties of $\varphi(x)$ under the modular group, it is now straightforward to determine the behavior of (1.6) as x approaches 1. This result of Ramanujan motivated the development of several methods to analyze (1.1); see, for example, Rademacher,(4) Knopp,(5) and Andrews.(6) One of the most elementary approaches is that of Kolberg,(7) which we employ here. Kolberg's method is summarized in Section 2 and applied to the $\mathbb{Z}_n$ model in Section 3 for the cases $n = 3$, 5, and 7. Some general remarks about the case when n is prime and of the form $6m + 1$ or $6m - 1$ are also given in Section 3. We note here that the critical exponent β is seen from these special cases to depend upon n.

2. KOLBERG'S METHOD

Let q denote any odd prime. Then for $s=0,\ldots, q-1$ define

$$g_s(x)=\sum_{(1/2)n(3n+1)\equiv s(\bmod q)} (-1)^n\, x^{n(3n+1)/2} \tag{2.1}$$

so that by a well-known identity of Euler we have

$$\varphi(x)=\sum_{s=0}^{q-1} g_s(x) \tag{2.2}$$

Kolberg[7] shows that if $24s+1$ is a quadratic nonresidue, then $g_s(x)=0$. (When $24s+1$ is a quadratic nonresidue there are no terms in (2.1).) Furthermore, if D denotes the determinant

$$D=\begin{vmatrix} g_0 & g_{q-1} & \cdots & g_1 \\ g_1 & g_0 & \cdots & g_2 \\ \vdots & \vdots & & \vdots \\ g_{q-1} & g_{q-2} & \cdots & g_0 \end{vmatrix} \tag{2.3}$$

and D_s is the determinant of the matrix formed by replacing the sth column of (2.3) by $(1, 0,\ldots, 0)^T$, then we have

$$P_s=\sum_{n=0}^{\infty} p(qn+s)\, x^{qn+s}=D_s/D \tag{2.4}$$

The reader can quickly verify this by multiplying the equation

$$\sum_{s=0}^{q-1} P_s=\varphi(x)^{-1}$$

by (2.2) and observing that $g_s P_{k-s}$ is x^k times some power series, thus giving q linear equations for P_s. Thus the evaluation of the sum in (1.1) is reduced to evaluating the functions $g_s(x)$ and the determinants in (2.4). Kolberg further shows that the determinant D can be expressed in terms of $\varphi(x)$

$$D=\varphi(x^q)^{q+1}/\varphi(x^{q^2}) \tag{2.5}$$

As it stands, the functions $g_s(x)$ as defined by (2.1) are not in a form suitable for analyzing the $x\to 1$ limit. For primes of the form $6m+1$ and $6m-1$, Atkin and Swinnerton-Dyer[8] have shown (Ref. 8, Lemma 6) how to express $g_s(x)$ in terms of the function

$$f(w, x)=\prod_{n=1}^{\infty} (1-x^{n-1}w)(1-x^n w^{-1})(1-x^n) \tag{2.6}$$

which is essentially an elliptic theta function. The cases used below ($q=5, 7$) go back to Watson. The function $f(w, x)$ frequently arises in exactly solvable models.[3] For us the most important property $f(w, x)$ satisfies is the transformation formula

$$\begin{aligned} &f(\exp(-2\pi z t), \exp(-2\pi t)) \\ &\quad = -t^{-1/2} \exp[\pi t/4 - \pi z(1-z)\, t]\, \vartheta_{11}(z, i/t) \end{aligned} \tag{2.7}$$

where $\vartheta_{11}(z, \tau)$ is the Jacobi theta function

$$\vartheta_{11}(z, \tau) = \sum_{n=-\infty}^{\infty} \exp[\pi i(n+1/2)^2 + 2\pi i(n+1/2)(z+1/2)] \tag{2.8}$$

A proof of (2.7) can be found in any textbook on theta functions or in Ref. 3 (see eq. 14.2.28). We will also need the transformation formula

$$\begin{aligned} &\varphi(\exp(-2\pi i/\tau)) \\ &\quad = \sqrt{-i\tau}\, \exp((\pi i/12\tau) + (\pi i\tau/12))\, \varphi(\exp(2\pi i\tau)) \end{aligned} \tag{2.9}$$

with $\mathrm{Im}(\tau) > 0$ and the Dedekind eta function

$$\eta(\tau) = \exp(\pi i\tau/12)\, \varphi(\exp(2\pi i\tau))$$

3. FORMULAS FOR $\mathrm{PROB}_q(\sigma = \omega^k)$

A. The $\mathbb{Z}_2$ Case

We first analyze the Onsager–Yang–Baxter result to clarify the notion of deviation from criticality as defined in the Introduction. For $n=2$ we have

$$\mathrm{Prob}_2(\sigma = 1) = (1 + \langle\sigma\rangle)/2 \tag{3.1}$$

where $\langle\sigma\rangle$ is the Onsager–Yang–Baxter spontaneous magnetization[3,9,10]

$$\langle\sigma\rangle = \prod_{n=1}^{\infty} \left(\frac{1 - x^{2n-1}}{1 + x^{2n-1}}\right) \tag{3.2}$$

First recall the product representations of the Jacobi theta functions

$$\begin{aligned} \vartheta_{00}(0, \tau) &= \prod_{n=1}^{\infty} (1 - x^{2n})(1 + x^{2n-1})^2 \\ \vartheta_{01}(0, \tau) &= \prod_{n=1}^{\infty} (1 - x^{2n})(1 - x^{2n-1})^2 \\ \vartheta_{10}(0, \tau) &= 2x^{1/4} \prod_{n=1}^{\infty} (1 - x^{2n})(1 + x^{2n})^2 \end{aligned} \tag{3.3}$$

where $x = \exp(\pi i \tau)$ and then write (3.2) as

$$\langle \sigma \rangle = \left(\frac{\vartheta_{01}(0, \tau)}{\vartheta_{00}(0, \tau)} \right)^{1/2} \tag{3.4}$$

The Jacobi theta functions $\vartheta_{00}(0, \tau)$ and $\vartheta_{01}(0, \tau)$ satisfy

$$\vartheta_{00}(0, -1/\tau) = (-i\tau)^{1/2}\, \vartheta_{00}(0, \tau)$$
$$\vartheta_{01}(0, -1/\tau) = (-i\tau)^{1/2}\, \vartheta_{10}(0, \tau)$$

so that

$$\begin{aligned}\langle \sigma \rangle &= \left(\frac{\vartheta_{10}(0, -1/\tau)}{\vartheta_{00}(0, -1/\tau)} \right)^{1/2} \\ &= 2^{1/2} p^{1/8} \prod_{n=1}^{\infty} \left(\frac{1 + p^{2n}}{1 + p^{2n-1}} \right) \qquad p = \exp(-\pi i/\tau)\end{aligned} \tag{3.5}$$

As $p \to 0$ we have

$$\text{Prob}_2(\sigma = 1) = \tfrac{1}{2} + 2^{-1/2} p^{1/8} + O(p^{9/8}) \tag{3.6}$$

so that $\beta = \frac{1}{8}$ is the critical exponent using p as the deviation from criticality variable.

B. The $\mathbb{Z}_3$ Case

For $q = 3$ we first evaluate the Kolberg determinants

$$\begin{aligned} D_0 &= g_0^2 - g_1 g_2 \\ D_1 &= g_2^2 - g_0 g_1 \\ D_2 &= g_1^2 - g_0 g_2 \end{aligned} \tag{3.7}$$

Using the Jacobi triple product identity Kolberg shows that

$$\begin{aligned} g_0(x) &= f(x^{12}, x^{27}) \\ g_1(x) &= -x f(x^6, x^{27}) \\ g_2(x) &= -x^2 f(x^3, x^{27}) \end{aligned} \tag{3.8}$$

Using (1.1), (2.5), (2.7), (3.7), and (3.8) we obtain

$$\begin{aligned} \text{Prob}_3(\sigma = 1) &= n_3(s)[\vartheta_{11}^2(4/9) - \vartheta_{11}(2/9)\, \vartheta_{11}(1/9)] \\ \text{Prob}_3(\sigma = \omega) &= n_3(s)[\vartheta_{11}^2(1/9) + \vartheta_{11}(2/9)\, \vartheta_{11}(4/9)] \\ \text{Prob}_3(\sigma = \omega^2) &= n_3(s)[\vartheta_{11}^2(2/9) + \vartheta_{11}(1/9)\, \vartheta_{11}(4/9)] \end{aligned} \tag{3.9}$$

where all ϑ functions are of modulus $\tau_3 = 2is/27$ and

$$\begin{aligned} n_3(s) &= 3^{-2} \exp(\pi s/27)\, \varphi(e^{-4\pi s})\, \varphi(e^{-4\pi s/9})/\varphi^4(e^{-4\pi s/3}) \\ &= 3^{-2}\eta(27\tau_3)\, \eta(3\tau_3)/\eta^4(9\tau_3) \end{aligned} \tag{3.10}$$

The small p expansion of (3.9) is now routine

$$\text{Prob}_3(\sigma = \omega^k) = \tfrac{1}{3} + c_k\, p^{4/27} + O(p^{8/27}) \tag{3.11}$$

where the constants c_k are given by

$$\begin{aligned} c_0 &= (4s_3/9)(s_1 + s_2 + 2s_4) = 1.1371580... \\ c_1 &= (-4s_3/9)(2s_1 + s_4 - s_2) = -0.3949308... \\ c_2 &= (-4s_3/9)(2s_2 - s_1 + s_4) = -0.742272... \end{aligned} \tag{3.12}$$

with $s_j = \sin(j\pi/9)$. Observe that the sum of these constants is zero reflecting that the sum of the probabilities is one. In the sense discussed in the Introduction, the critical exponent β for the $\mathbb{Z}_3$ Baxter model is 4/27.

C. The $\mathbb{Z}_5$ Case

For $q = 5$ we have

$$\begin{aligned} D_0 &= g_0^4 - g_1^3 g_2 + 2g_0 g_1 g_2^2 \\ D_1 &= -g_0^3 g_1 - g_0 g_2^3 + g_1^2 g_2^2 \\ D_2 &= -g_0^3 g_2 + g_0^2 g_1^2 - g_1 g_2^3 \\ D_3 &= 2g_0^2 g_1 g_2 - g_0 g_1^3 + g_2^4 \\ D_4 &= g_0^2 g_2^2 - 3g_0 g_1^2 g_2 + g_1^4 \end{aligned} \tag{3.13}$$

From Atkin and Swinnerton-Dyer[8] we find that

$$\begin{aligned} g_0(x) &= f^2(x^{10}, x^{25})/\varphi(x^5) \\ g_1(x) &= -xf(x^{10}, x^{25})\, f(x^5, x^{25})/\varphi(x^5) \\ g_2(x) &= -x^2 f^2(x^5, x^{25})/\varphi(x^5) \end{aligned} \tag{3.14}$$

Proceeding as in the above $q = 3$ case we conclude

$$\begin{aligned} \text{Prob}_5(\sigma = 1) &= n_5(s)\, \vartheta_{11}^3(2/5)[\vartheta_{11}^5(2/5) - 3\vartheta_{11}^5(1/5)] \\ \text{Prob}_5(\sigma = \omega) &= n_5(s)\, \vartheta_{11}(1/5)\, \vartheta_{11}^2(2/5)[\vartheta_{11}^5(2/5) + 2\vartheta_{11}^5(1/5)] \\ \text{Prob}_5(\sigma = \omega^2) &= n_5(s)\, \vartheta_{11}^2(1/5)\, \vartheta_{11}(2/5)[2\vartheta_{11}^5(2/5) - \vartheta_{11}^5(1/5)] \\ \text{Prob}_5(\sigma = \omega^3) &= n_5(s)\, \vartheta_{11}^3(1/5)[3\vartheta_{11}^5(2/5) + \vartheta_{11}^5(1/5)] \\ \text{Prob}_5(\sigma = \omega^4) &= n_5(s)\, 5\vartheta_{11}^4(2/5)\, \vartheta_{11}^4(1/5) \end{aligned} \tag{3.15}$$

where all theta functions are modulus $\tau_5 = 2is/25$ and

$$n_5(s) = 5^{-4} \exp(4\pi s/25) \frac{\varphi(e^{-4\pi s})\,\varphi(e^{-(4\pi/25)s})}{\varphi^{10}(e^{-4\pi s/5})}$$

$$= 5^{-4}\eta(25\tau_5)\,\eta(\tau_5)/\eta^{10}(5\tau_5) \tag{3.16}$$

Using the identity

$$\vartheta_{11}(1/5)\,\vartheta_{11}(2/5) = \sqrt{5}\,q^{1/2}\varphi(q^{10})\,\varphi(q^2), \qquad q = \exp(\pi i\tau_5)$$

the $\mathrm{Prob}_5(\sigma = \omega^4)$ simplifies to

$$\mathrm{Prob}_5(\sigma = \omega^4) = 5^{-1}\varphi(q^{50})\,\varphi^5(q^2)/\varphi^6(q^{10})$$

which is essentially the Ramanujan result (1.6) in conjugate variables. The small p expansion of (3.15) is straightforward, although rather messy

$$\mathrm{Prob}_5(\sigma = \omega^k) = \tfrac{1}{5} + c_k\,p^{4/25} + c_k'\,p^{8/25} + O(p^{12/25}) \tag{3.17}$$

for $k = 0, 3, 4$ and

$$\mathrm{Prob}_5(\sigma = \omega^k) = \tfrac{1}{5} + c_k\,p^{8/25} + O(p^{12/25})$$

for $k = 1, 2$ where the constants are given by

$$\begin{aligned}
c_0 &= (\sqrt{5}+1)/2 \qquad c_0' = 0\\
c_1 &= (\sqrt{5}-1)/2,\\
c_2 &= -(\sqrt{5}+1)/2,\\
c_3 &= -(\sqrt{5}-1)/2 \qquad c_3' = 0\\
c_4 &= -1,\ c_4' = 1
\end{aligned} \tag{3.18}$$

Thus the exponent β is 4/25 for $k = 0, 3, 4$ and 8/25 for $k = 1, 2$.

D. The $\mathbb{Z}_7$ Case

The determinants D_s, $s = 0,\ldots, 6$ are rather involved and will not be explicitly written out here (each D_s is a homogeneous polynomial in g_0, g_1, g_2, and g_5 of degree 6 with 12 nonzero terms). Again product representations for the nonzero g_s's can be found in Ref. 8. In terms of $f(z, x)$ they are given by

$$\begin{aligned}
g_0 &= f_2^2 f_3/[\varphi(y)\,\varphi(y^7)]\\
g_1 &= -x f_1 f_3^2/[\varphi(y)\,\varphi(y^7)]\\
g_2 &= -x^2 f_1 f_2 f_3/[\varphi(y)\,\varphi(y^7)]\\
g_5 &= x^5 f_1^2 f_2/[\varphi(y)\,\varphi(y^7)]
\end{aligned} \tag{3.19}$$

where we have used the abbreviation $f_j = f(x^{7j}, x^{49})$ and $y = x^7$. The resulting formulas for D_s can be further simplified if we make use of the fact that there are identities connecting the various functions $g_s(x)$. Kolberg derives these using the Jacobi identity for $\varphi(x)^3$. In terms of the functions f_j, the identity we will use becomes

$$y f_1^3 f_2 = -f_1 f_3^3 + f_2^3 f_3 \tag{3.20}$$

We proceed as above: first, express D_s in terms of the functions f_j using (3.19); second, eliminate all terms with a "y variable" using the identity (3.20), and third, use (2.7) and (2.9). The result is

$$\begin{aligned}
\mathrm{Prob}_7(\sigma = 1) &= n_7(s)[-13 t_1 t_2^9 t_3^8 - t_1^2 t_2^6 t_3^{10} \\
&\quad + 2 t_1^3 t_2^3 t_3^{12} + 11 t_2^{12} t_3^6 + 2 t_1^4 t_3^{14}] \\
\mathrm{Prob}_7(\sigma = \omega) &= n_7(s)[15 t_1 t_2^{10} t_3^7 - 15 t_1^2 t_2^7 t_3^9 \\
&\quad + 9 t_1^3 t_2^4 t_3^{11} - 5 t_1^4 t_2 t_3^{13} - 3 t_2^{13} t_3^5] \\
\mathrm{Prob}_7(\sigma = \omega^2) &= n_7(s)[-5 t_1 t_2^{11} t_3^6 + 26 t_1^2 t_2^8 t_3^8 - 31 t_1^3 t_2^5 t_3^{10} \\
&\quad + 11 t_1^4 t_2^2 t_3^{12} + t_2^{14} t_3^4] \\
\mathrm{Prob}_7(\sigma = \omega^3) &= n_7(s)[5 t_1 t_2^{12} t_3^5 - 11 t_1^2 t_2^9 t_3^7 \\
&\quad + 18 t_1^3 t_2^6 t_3^9 - 8 t_1^4 t_2^3 t_3^{11} - t_1^5 t_3^{13}] \\
\mathrm{Prob}_7(\sigma = \omega^4) &= n_7(s)[-t_1 t_2^{13} t_3^4 + 12 t_1^2 t_2^{10} t_3^6 - 12 t_1^3 t_2^7 t_3^8 \\
&\quad + 3 t_1^4 t_2^4 t_3^{10} + 3 t_1^5 t_2 t_3^{12}] \\
\mathrm{Prob}_7(\sigma = \omega^5) &= n_7(s)\, 7 t_1^2 t_2^2 t_3^2 [2 t_1 t_2^6 t_3^5 + t_1^2 t_2^3 t_3^7 \\
&\quad - t_1^3 t_3^9 - t_2^9 t_3^3] \\
\mathrm{Prob}_7(\sigma = \omega^6) &= n_7(s)\, t_1^2 t_3 [-10 t_1 t_2^9 t_3^5 + 17 t_1^2 t_2^6 t_3^7 \\
&\quad + t_1^3 t_2^3 t_3^9 + t_1^4 t_3^{11}]
\end{aligned} \tag{3.21}$$

where

$$t_j = \vartheta_{11}(j/7, \tau_7), \qquad j = 1, 2, 3 \qquad \text{with} \qquad \tau_7 = 2is/49 \qquad \text{and}$$

$$\begin{aligned}
n_7(s) &= 7^{-6} \exp\left(\frac{9\pi}{49} s\right) \frac{\varphi(e^{-4\pi s})}{\varphi^{14}(e^{-4\pi s/7})\, \varphi^5(e^{-4\pi s/49})} \\
&= 7^{-6} \frac{\eta(49\tau_7)}{\eta^{14}(7\tau_7)\, \eta^5(\tau_7)}
\end{aligned}$$

The expression for $\mathrm{Prob}_7(\sigma = \omega^5)$ can be further simplified but we do not pursue this here.[5] From the above expressions it is now clear that the correction terms to the constant 1/7 are of order $p^{4/49}$ times some constant. However, an evaluation of these constants shows that they are all zero. A calculation of the next term which is of order $p^{8/49}$ shows that the constants are nonzero only in the cases $k = 0, 2, 5$, and 6. If c is the constant for $k = 0$, then $1 - (1/c)$ is the constant for $k = 2$, and $(1/c) - c$ is the constant for $k = 6$. The constant for $k = 5$ is -1. These results are based upon a numerical evaluation of the constants expressed in terms of the relevant trigonometric functions. Numerically we find $c = 1.80193773580485...$. We have not investigated the higher-order terms.

E. The $\mathbb{Z}_q$ Case, q Prime and of Form $6m+1$ or $6m-1$

The Kolberg method works for any odd prime q and the functions $g_s(x)$ have by Lemma 6 of Ref. 8 product expansions expressible in terms of $f(w, y)$ where $y = x^q$ and $w = x^{jq}$, $j = 0,..., q-1$ whenever q is of the form $6m+1$ or $6m-1$. Thus the general structure of the sum (1.1) will be similar to the above cases $q = 5, 7$ and the correction term to the leading $1/q$ behavior will be of order p^{4/q^2}. It may happen that the constant multiplying p^{4/q^2} vanishes, in which case the correction term will be higher-order, say p^{8/q^2}.

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